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ABSTRACT

Early results are reported for an effort to use a reasoning-based mathematics program in the primary classroom. In the preschool years, children develop knowledge about quantities of material, referred to as protoquantitative reasoning. Integrating counting with the protoquantitative schemas seems to occur as early as 4 years of age. Drawing on this knowledge of children's abilities, the following set of principles was developed for the primary arithmetic program: (1) develop children's trust in their own knowledge; (2) draw children's informal knowledge into the classroom; (3) use formal notation as a record of discussions and conclusions; (4) introduce key mathematical structures as quickly as possible; (5) encourage everyday problem finding/solving; and (6) talk about mathematics, rather than just doing arithmetic. These principles have been applied to two cohorts of children, one beginning in the first grade, and the other beginning in the second grade. In these two classes, the program appeared effective in raising mathematics scores on the California Achievement Test for children of all ability levels, and the approach was successful for educationally disadvantaged children. The program is described as a version of the cognitive apprenticeship called for by A. Collins and others in an influential paper (1989). These initial efforts will be part of a long-term project to revise instructional practice. Seven figures and a 12-item list of references are included. The paper's discussant is Nancy Vye (and others) in a taining piece entitled "Exploring a Reasoning-Based Program for Teaching Arithmetic". (SLD)

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# THINKING IN ARITHMETIC CLASS

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## THINKING IN ARITHMETIC CLASS

For many years now, most efforts to improve educational outcomes for disadvantaged students have been based on the premise that what such children need is higher expectations for learning coupled with intensified and careful application of traditional classroom methods. Thus, what is typically prescribed is more careful explanations, more practice, and more frequent testing to monitor progress. Such methods seem to work—up to a point. That is, they produce gains on basic skills tests. But they are not designed to teach children to reason and solve problems. Today, such abilities are fundamental for participation in the economy and society in general.

The nearly exclusive focus on the kinds of "basic skills" that can be taught by repetitive drill does not necessarily derive from a lack of ambition for disadvantaged students or from a belief that the children are inherently incapable of thinking and problem solving. Rather, it is rooted in an assumption that most educators share about all learning by nearly all children (some would except the "gifted"): that successful learning means working step by step through a hierarchical sequence of skills and concepts. The common view is that skills and concepts are ordered in rather strict hierarchies, and that asking children to perform complex skills before they master the prerequisite, simpler ones is to doom them to failure, or at least to frustration, in the course of learning. This hierarchical mastery learning approach dictates that children who have trouble learning some of the simpler skills practice them longer. But in practice this turns out to deny disadvantaged children the opportunity to learn higher-order abilities. Because many disadvantaged are among those who learn slowly at the outset, they are doomed to more and more supervised practice on the "basics." They never get to graduate to the more demanding and interesting problems that constitute the "higher-order" part of the curriculum.

The work we describe in this paper is premised on a radically different set of assumptions. We argue that disadvantaged children, like all children, can begin their educational life by engaging in active thinking and problem solving. We argue further that, when thinking-oriented instruction is carefully organized for this purpose, children can acquire the traditional basic skills in the process of reasoning and solving problems. As a result, they can acquire not only the fundamentals of a discipline, but also the ability to apply those fundamentals, and—critically—a belief in their own capacities as learners and thinkers.

Reviewing research and practical efforts to teach higher-order thinking skills a few years ago, Resnick (1987) concluded that shaping a disposition to critical thought is as important in developing higher-order cognitive abilities in students as is teaching particular skills of reasoning and thinking. Acquiring such dispositions, she proposed, requires regular participation in activities that exercise reasoning skills within social

environments that value thinking and judgment and that communicate to children a sense of their own competence in reasoning and thinking. This, in turn, calls for educational programs suffused with thinking and reasoning, programs in which basic subject matter instruction serves as the daily occasion for exercising and extending cognitive abilities. Explicit attention to thinking and reasoning seems particularly important for children who are not experiencing daily practice in such reasoning in their homes or who do not trust their own out-of-school experience as being relevant to school success. Such children often fail to learn the "hidden curriculum" of thinking and reasoning that more favored children acquire without much explicit help from teachers.

We report here on the early results of an effort to apply these ideas to early mathematics teaching for disadvantaged children. To embed basic mathematics learning in a thinking curriculum, we had to design a new set of practices for the mathematics classroom. We wanted to create an environment in which children would practice mathematics as a field in which there are open questions and arguments, in which interpretation, reasoning, and debate—all key components of critical thought—play a legitimate and expected role. To do this, we needed to revise mathematics teaching in the direction of treating mathematics as if it were an ill-structured discipline. That is, we needed to take seriously, with and for young learners, the propositions that mathematical statements can have more than one interpretation, that interpretation is the responsibility of every individual using mathematical expressions, and that argument and debate about interpretations and their implications are a normal part of mathematical activity. Participating in such an environment, we thought, would develop capabilities and dispositions for finding relationships among mathematical ideas and between mathematical statements and problem situations. It would develop skill not only in applying mathematics but also in thinking mathematically. In short, it would socialize children into a developmentally appropriate form of the practice of mathematics as a mode of thought, reasoning, and problem solving.

This goal, however, seemed at first to pose an insurmountable problem for school beginners—especially, perhaps, those we label *disadvantaged*. To engage in the kind of mathematical discussions we were aiming for, children would have to know *some* mathematics at the outset. They would need something to think *about* if the exercise was not to be an empty one. A first question, then, was whether children entering school knew enough about numbers and quantities to permit a reasoning- and discussion-oriented program from the outset. Fortunately, a large body of research accumulated over the past decade suggests that almost all children come to school with a substantial body of knowledge about quantity relations and that children are capable of using this knowledge as a foundation for understanding numbers and arithmetic (see Resnick, 1989; Resnick & Greeno, 1990, for interpretive reviews). This knowledge, we thought, could provide the initial foundations for children's participation in a reasoning-based mathematics program.

## The Intuitive Basis for Early Mathematical Reasoning

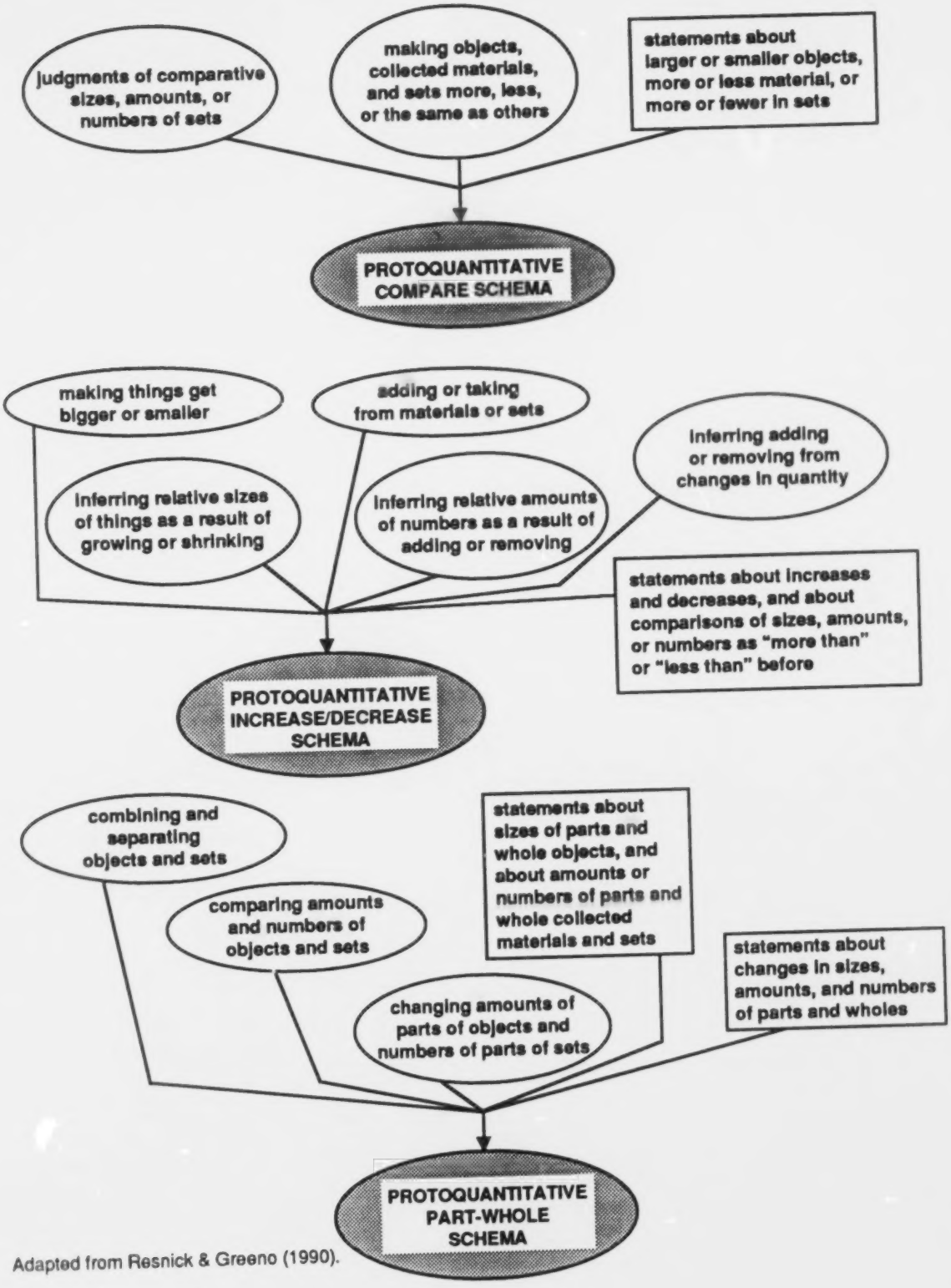
Children come to school with two kinds of intuitively developed knowledge relevant to mathematics learning. First, they know a good deal about amounts of physical material and the relations among these amounts, even though they cannot yet use numbers to describe these relations. Second, most children know the rules for counting sets of objects. This gives them the beginning tool for using numbers to manipulate and describe quantity relations.

### Protoquantitative Schemas

During the preschool years, children develop a large store of knowledge about how quantities of physical material behave in the world. This knowledge, acquired from manipulating and talking about physical material, allows children to compare amounts and sizes and to reason about changes in amounts and quantities. Because this early reasoning about amount of material is done without measurement or exact numerical quantification, we refer to it as *proto*quantitative reasoning. We can document development during the preschool years of three sets of protoquantitative schemas: *compare*, *increase/decrease*, and *part-whole* (see Figure 1).

The *protoquantitative compare* schema makes greater-smaller comparative judgments of amounts of material. Before they are two years old, children express quantity judgments in the form of absolute size labels such as *big*, *small*, *lots*, and *little*. Only a little later, they begin to put linguistic labels on the comparisons of sizes they made as infants. Thus, they can look at two circles and declare one bigger than the other, see two trees and declare one taller than the other, examine two glasses of milk and declare that one contains more than the other. These comparisons initially are based on direct perceptual judgments without any measurement process. However, they form a basis for eventual numerical comparisons of quantity.

The *protoquantitative increase/decrease* schema interprets changes as increases or decreases in quantities. This schema allows children as young as three or four years of age to reason about the effects of adding or taking away an amount from a starting amount. Children know, for example, that if they have a certain amount of something and they get another amount of the same thing (perhaps mother adds another cookie to the two already on the child's plate), they have more than before. Or, if some of the original quantity is taken away, they have less than before. Equally important, children know that if nothing has been added or taken away, they have the same amount as before. For example, children show surprise and label as "magic" any change in the number of objects on a plate that occurs out of their sight (Gelman, 1972). This shows that children have the underpinnings of number conservation well before they can pass the standard Piagetian tests. They can be fooled by perceptual cues or language that distracts them from quantity, but they possess a basic understanding of addition,


judgments of comparative sizes, amounts, or numbers of sets
making objects, collected materials, and sets more, less, or the same as others
statements about larger or smaller objects, more or less material, or more or fewer in sets
PROTOQUANTITATIVE COMPARE SCHEMA
making things get bigger or smaller
adding or taking from materials or sets
inferring adding or removing from changes in quantity
inferring relative sizes of things as a result of growing or shrinking
inferring relative amounts of numbers as a result of adding or removing
statements about increases and decreases, and about comparisons of sizes, amounts, or numbers as "more than" or "less than" before
PROTOQUANTITATIVE INCREASE/DECREASE SCHEMA
combining and separating objects and sets
statements about sizes of parts and whole objects, and about amounts or numbers of parts and whole collected materials and sets
comparing amounts and numbers of objects and sets
statements about changes in sizes, amounts, and numbers of parts and wholes
changing amounts of parts of objects and numbers of parts of sets
PROTOQUANTITATIVE PART-WHOLE SCHEMA

Adapted from Resnick & Greeno (1990).

**FIGURE 1 THE PROTOQUANTITATIVE SCHEMAS**

subtraction, and conservation. The protoquantitative increase/decrease schema is also the foundation for eventual understanding of unary addition and subtraction.

The *protoquantitative part-whole* schema is really a set of schemas that organize children's knowledge about the ways in which material around them comes apart and goes together. The schemas specify that material is *additive*. That is, one can cut a quantity into pieces that, taken together, equal the original quantity. One can also put two quantities together to make a bigger quantity and then join that bigger quantity with yet another in a form of hierarchical additivity. Implicitly, children know about this additive property of quantities. This protoquantitative knowledge allows them to make judgments about the relations between parts and wholes, including class inclusion (Markman & Siebert, 1976) and the effects of changes in the size of parts on the size of the whole. The protoquantitative part-whole schema is the foundation for later understanding of binary addition and subtraction and for several fundamental mathematical principles, such as the commutativity and associativity of addition and the complementarity of addition and subtraction. It also provides the framework for a concept of additive composition of number that underlies the place value system.

### Counting

Counting is the first step in making quantitative judgments exact. It is a measurement system for sets. Gelman and her colleagues have shown that children as young as three or four years of age implicitly know the key principles that allow counting to serve as a vehicle of quantification (Gelman & Gallistel, 1978). These principles include the knowledge that number names must be matched one-for-one with the objects in a set and that the order of the number names matters, but the order in which the objects are touched does not. Knowledge of these principles is inferred from the ways in which children solve novel counting problems. For example, if asked to make the second object in a row "number 1," children do not neglect the first object entirely but, rather, assign it one of the higher number names in the sequence.

Other research has challenged Gelman's assessment of the ages at which children can be said to have acquired all of the counting principles. Some of the challenges are really arguments about the criteria for applying certain terms. For example, Gelman has attributed knowledge of *cardinality*, a key mathematical principle, to children as soon as they know that the last number in a counting sequence names the quantity in the whole set; others would reserve the term for a more advanced stage in which children reliably conserve quantity under perceptual transformations. A challenge that goes beyond matters of terminology comes from research showing that, although children may know all the principles of counting and be able to use counting to quantify given sets of objects or to create sets of specified sizes, they may not, at a certain point, have fully integrated their counting knowledge with their protoquantitative knowledge. Several investigators (e.g., Sophian, 1987) have shown that many children who know how to count sets do not



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spontaneously count in order to compare sets. This means that counting and the protoquantitative schemas exist initially as separate knowledge systems, isolated from each other.

***Integrating counting with the protoquantitative schemas.*** Such findings make it clear that, even after knowledge of counting principles is established, there is substantially more growth in number concepts still to be attained. A first major step in this growth is integration of the number-name sequence with the protoquantitative comparison schema. This seems to happen as young as about four years of age. At this point, children behave as if the counting word sequence constitutes a kind of "mental number line" (Resnick, 1983). They can quickly identify which of a pair of numbers is more by mentally consulting this number line, without actually stepping through the sequence to determine which number comes later.

In the child's subsequent development, counting as a means of quantifying sets is integrated with the protoquantitative part-whole and increase/decrease schemas. This integration seems to develop as a result of participating in situations in which changes and combinations of quantity are called for and there is a cultural mandate for exact quantification. Out of school, this can occur in various play or household activities—particularly when age segregation is not strict so that young children engage freely with older children and adults. School settings can mimic the conditions of everyday life to some extent. However, a principal resource for promoting quantification of the schemas in school is the story problem. Several researchers (e.g., De Corte & Verschaffel, 1987; Riley & Greeno, 1988) have shown that children entering school can solve many simple story problems by applying their counting skills to sets they create as they build physical models of the story situations. Because the stories involve the same basic relationships among quantities as the protoquantitative schemas, extensive practice in solving problems via counting should help children quantify their original schemas. Such practice should not only develop children's ability to solve problems using exact numerical measures, but also lead them to interpret numbers themselves in terms of the relations specified by the protoquantitative schemas. Eventually, children should be able to construct an enriched meaning for numbers—treating numbers (rather than measured quantities of material) as the entities that are mentally compared, increased and decreased, or organized into parts and wholes.

## Principles for a Reasoning-Based Arithmetic Program

With this research base as a grounding for our efforts, we set out to develop a primary arithmetic program (for grades 1 through 3) that would engage children from the outset in invention, reasoning, and verbal justification of mathematical ideas. The school in which we worked served mainly minority (94% were African-Americans), low-income (69% were eligible for free or reduced-price lunches) children. Our goal was to use as little traditional school drill material as possible in order to provide for children a

consistent environment in which they would be socialized to think of themselves as mathematical reasoners and to behave accordingly. This meant that we needed a program in which children would successfully learn the traditional basics of arithmetic calculation as well as more complex forms of reasoning and argumentation. The program evolved gradually over a period of months. We describe it here in somewhat schematized form as the instantiation of a set of six principles that guided our thinking and experimentation.

1. ***Develop children's trust in their own knowledge.*** Traditional instruction, by focusing on specific procedures and on special mathematical notations and vocabulary, tends to teach children that what they already know is not legitimately mathematics. To develop children's trust in their own knowledge as mathematics, our program stresses the possibility of multiple procedures for solving any problem, invites children's invention of these multiple procedures, and asks that children explain and justify their procedures using everyday language. In addition, the use of manipulatives and finger counting ensures that children have a way of establishing for themselves the truth or falsity of their proposed solutions. Figure 2 provides examples of multiple procedures used by second-grade children to solve the same addition problem, *158 + 74*. The examples are copied from six different children's homework papers. Child A used a procedure of adding the value of the leftmost digits, first *100 + 70*, then *50 + 4*. This unusual decomposition left the *8* of *158* still to be added, which the child added to the already accumulated *54*. To add the resulting *62* to *170*, the child decomposed it to *60* and *2*. He added to *60* first, yielding *230*, and then the *2*, to yield the final answer. Child F used a more conventional place value decomposition, first adding up the hundreds (note that she indicates that there are *0* hundreds in *74*), then the tens, then the units, and finally combining the three partial sums. Child E also used a place value decomposition but worked initially on the hundreds and tens combined (*150 + 70*). These and the other solutions in the figure illustrate the ways in which written notation and mental arithmetic are combined in the children's procedures.

***2. Draw children's informal knowledge, developed outside school, into the classroom.*** An important early goal of the program is to stimulate the use of counting in the context of the compare, increase/decrease, and part-whole schemas to promote children's construction of the quantified versions of those schemas. This is done through extensive problem-solving practice, using both story problems and acted-out situations. Counting (including counting on one's fingers) is actively encouraged. Figure 3 gives an example of a typical class problem, showing how it can generate several solutions; the notations shown are copied from the notebook in which a child recorded the solutions proposed by several teams who had worked on the problem.

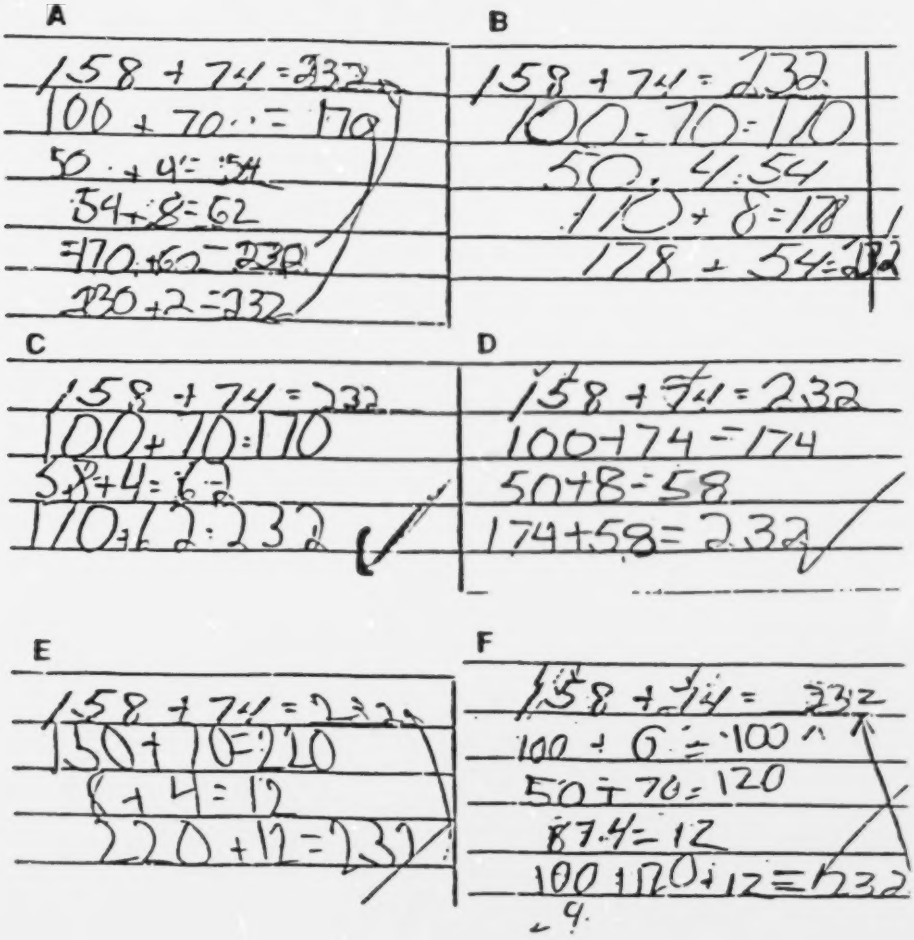


**FIGURE 2 EXAMPLES OF SEVERAL SECOND-GRADERS' SOLUTIONS TO A COMPUTATIONAL PROBLEM**

Mary told her friend Tonya that she would give her 95 barrettes. Mary had 4 bags of barrettes and each bag had 9 barrettes. Does Mary have enough barrettes?

The class first developed an estimated answer. Then they were asked, "How many more does she need?" The solutions below were generated by different class groups.

Group 1 first solved for the number of barrettes by repeated addition. Then they decomposed 4 x 9 into 2 x 9 plus 2 x 9. Then they set up a missing addend problem, 36 + 59, which they solved by a combination of estimation and correction.

Group 2 set up a subtraction equation and then developed a solution that used a negative partial result.

Group 4 began with total number of barrettes needed and subtracted out the successive bags of 9.

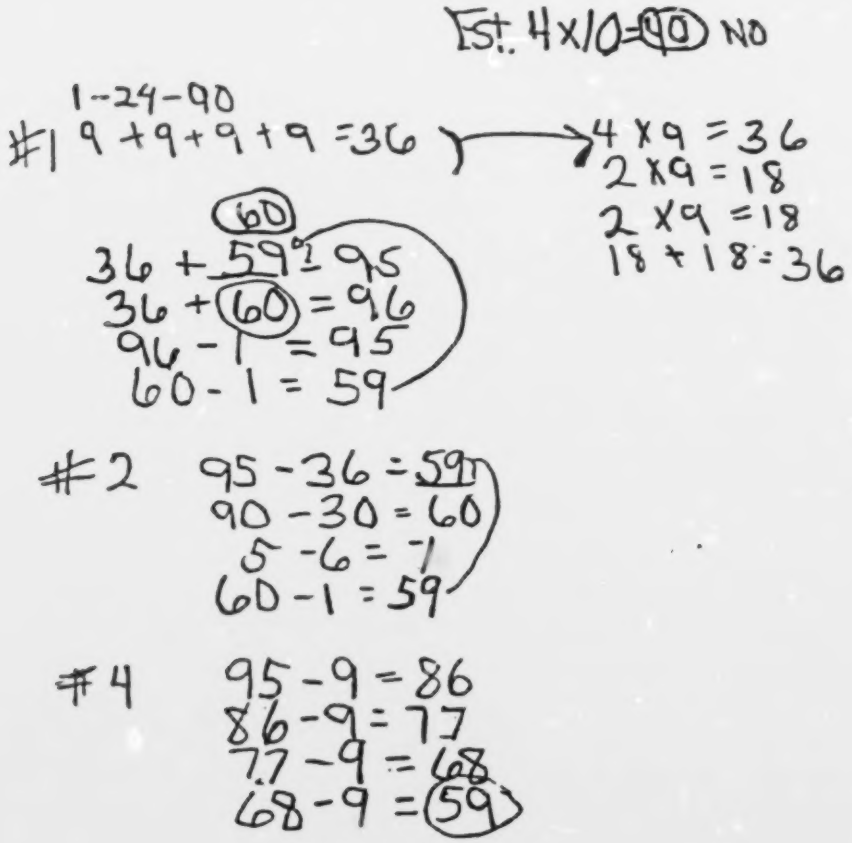


**FIGURE 3 A SECOND-GRADE PROBLEM AND SEVERAL SOLUTIONS**

***3. Use formal notations (identity sentences and equations) as a public record of discussions and conclusions.*** Children's intuitive knowledge must be linked to the formal language of mathematics. By using a standard mathematical notation to record conversations carried out in ordinary language and rooted in well-understood problem situations, the formalisms take on a meaning directly linked to children's mathematical intuitions. First used by the teacher as a way of displaying for the class what a child had proposed, equations quickly became common currency in the classroom. Most of the children began to write equations themselves only a few weeks into the school year. Figure 4 shows part of a typical teacher-led sequence in which children propose a solution to a story problem. The teacher carefully linked elements of the proposed solution to the actual physical material involved in the story (the tray of cupcakes) and an overhead schematic of the material. Only after the referential meaning of each number had been carefully established was the number written into the equation. The total sequence shown took about 1 minute 20 seconds.

***4. Introduce key mathematical structures as quickly as possible.*** Children's protoquantitative schemas already allow them to think reasonably powerfully about how amounts of material compare, increase and decrease, come apart and go together. In other words, they already know, in nonnumerically quantified form, something about properties such as commutativity, associativity, and additive inverse. A major goal of the first year or two of school mathematics is to "mathematize" this knowledge—that is, quantify it and link it to formal expressions and operations. It was our conjecture that this could best be done by laying out the additive structures (e.g., for first grade: addition and subtraction problem situations, the composition of large numbers, regrouping as a special application of the part-whole schemas) as quickly as possible and then allowing full mastery (speed, flexibility of procedures, articulate explanations) of elements of the system to develop over an extended time. Guided by this principle, we found it possible to introduce addition and subtraction with regrouping in February of first grade. However, no specific procedures were taught; rather, children were encouraged to invent (and explain) ways of solving multidigit addition and subtraction problems, using appropriate manipulatives and/or expanded notation formats that they developed.

It is important to note that a program built around this principle constitutes a major challenge to an idea that has been widely accepted in the past twenty or thirty years of educational research and practice. This is the notion of learning hierarchies—specifically, that it is necessary for learners to master simpler components before they try to learn complex skills. According to theories of hierarchical and mastery learning, children should thoroughly master single-digit addition and subtraction, for example, before attempting multidigit procedures, and they should be able to perform multidigit arithmetic without regrouping smoothly before they tackle the complexities of regrouping. We propose instead a *distributed* curriculum in which multiple topics are developed all year long, with increasing levels of sophistication and demand, rather than a strictly sequential curriculum.

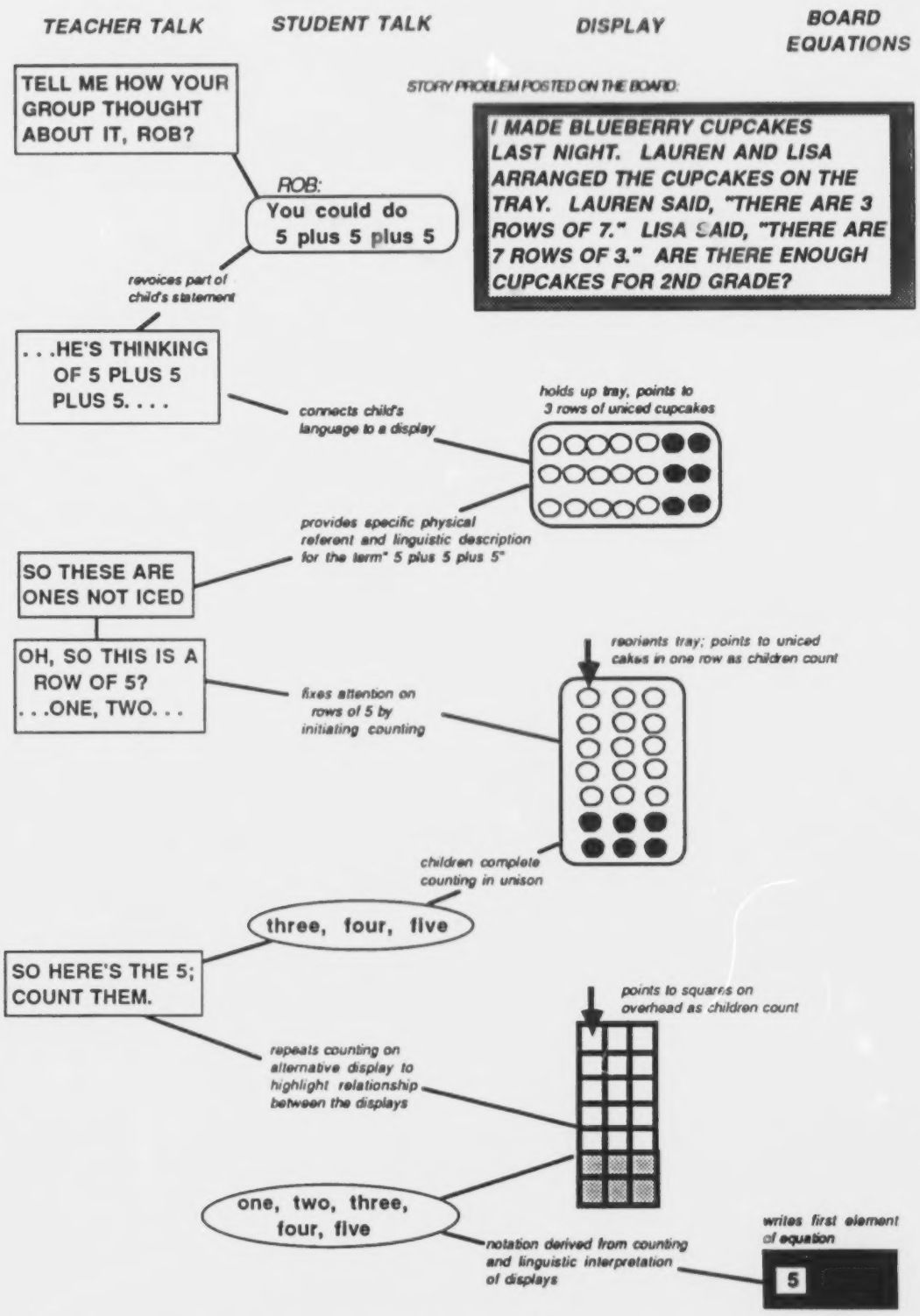
TEACHER TALK
STUDENT TALK
DISPLAY
BOARD EQUATIONS
TELL ME HOW YOUR GROUP THOUGHT ABOUT IT, ROB?
STORY PROBLEM POSTED ON THE BOARD:
I MADE BLUEBERRY CUPCAKES LAST NIGHT. LAUREN AND LISA ARRANGED THE CUPCAKES ON THE TRAY. LAUREN SAID, "THERE ARE 3 ROWS OF 7." LISA SAID, "THERE ARE 7 ROWS OF 3." ARE THERE ENOUGH CUPCAKES FOR 2ND GRADE?
ROB:
You could do 5 plus 5 plus 5
revoices part of child's statement
. . .HE'S THINKING OF 5 PLUS 5 PLUS 5. . . .
connects child's language to a display
holds up tray, points to 3 rows of uniced cupcakes
provides specific physical referent and linguistic description for the term" 5 plus 5 plus 5"
SO THESE ARE ONES NOT ICED
OH, SO THIS IS A ROW OF 5? . . .ONE, TWO. . .
fixes attention on rows of 5 by initiating counting
reorients tray; points to uniced cakes in one row as children count
children complete counting in unison
three, four, five
SO HERE'S THE 5; COUNT THEM.
repeats counting on alternative display to highlight relationship between the displays
points to squares on overhead as children count
one, two, three, four, five
notation derived from counting and linguistic interpretation of displays
writes first element of equation
5

**FIGURE 4 PART OF A WHOLE-CLASS DISCUSSION OF A STORY PROBLEM**



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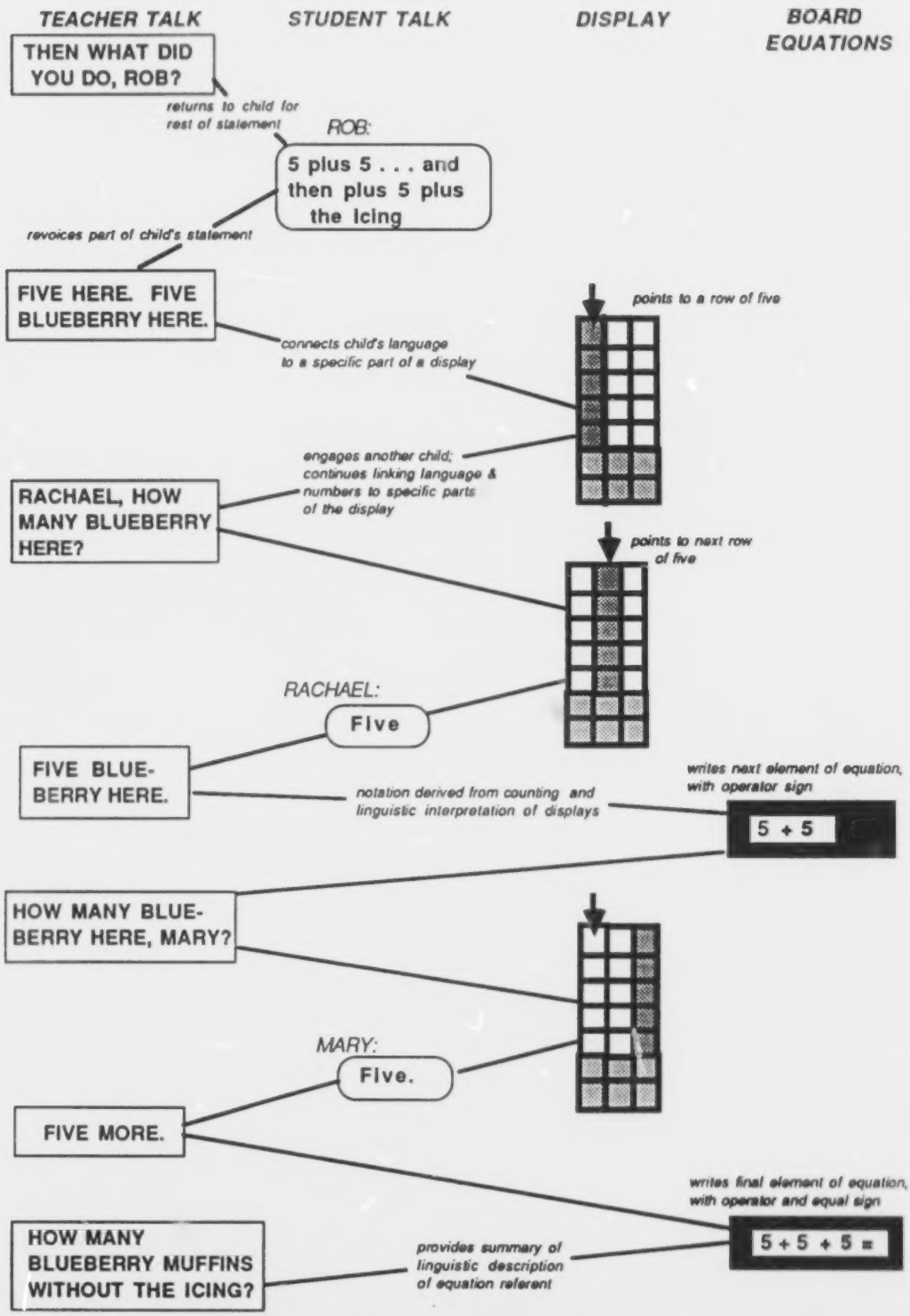

TEACHER TALK
STUDENT TALK
DISPLAY
BOARD EQUATIONS
THEN WHAT DID YOU DO, ROB?
returns to child for rest of statement
ROB:
5 plus 5 . . . and then plus 5 plus the Icing
revoices part of child's statement
FIVE HERE. FIVE BLUEBERRY HERE.
connects child's language to a specific part of a display
points to a row of five
engages another child; continues linking language & numbers to specific parts of the display
RACHAEL, HOW MANY BLUEBERRY HERE?
points to next row of five
RACHAEL:
Five
FIVE BLUE-BERRY HERE.
notation derived from counting and linguistic interpretation of displays
writes next element of equation, with operator sign
5 + 5
HOW MANY BLUE-BERRY HERE, MARY?
MARY:
Five.
FIVE MORE.
writes final element of equation, with operator and equal sign
5 + 5 + 5 =
HOW MANY BLUEBERRY MUFFINS WITHOUT THE ICING?
provides summary of linguistic description of equation referent

**FIGURE 4 PART OF A WHOLE-CLASS DISCUSSION OF A STORY PROBLEM (Concluded)**

To convey the flavor of the process, Figure 5 shows the range of topics planned for a single month of the second-grade program. All topics shown are treated at changing levels of sophistication and demand throughout the school year. This distributed curriculum discourages decontextualized teaching of components of arithmetic skill. It encourages children to draw on their existing knowledge framework (the protoquantitative schemas) to interpret advanced material, while gradually building computational fluency.

| Domain | Specific Content |
|---|---|
| Reading/Writing Numerals | 0-9,999 |
| Set Counting | 0-9,999 |
| Addition | 2- and 3-digit regrouping, Basic Facts 20 |
| Subtraction | 2-digit renaming, Basic Facts 20 |
| Word Problems | Addition, Subtraction, Multiplication |
| Problem Solving | Work backward, Solve an easier problem, Patterns |
| Estimation | Quantities, Strategies, Length |
| Ratio/Proportion | Scaling up, Scaling down |
| Statistics/Probability | Scaling up, Scaling down, Spinner (1/4), Dice (1/16), 3 graphs |
| Multiplication | Array (2, 4 tables), Allocation, Equal groupings |
| Division | Oral problems involving sharing sets equally |
| Measurement | Arbitrary units |
| Decimals | Money |
| Fractions | Parts of whole, Parts of set, Equivalent pieces |
| Telling Time | To hour, To half hour |
| Geometry | Rectangle, square (properties) |
| Negative Integers | Ones, tens |

**FIGURE 5 TOPIC COVERAGE PLANNED FOR A SINGLE MONTH OF GRADE 2**

***5. Encourage everyday problem finding.*** In stating this principle, we deliberately use the term *everyday* in two senses. First, it means literally doing arithmetic every day, not only in school but also at home and in other informal settings. Children need massive practice in applying arithmetic ideas, far more than the classroom itself can provide. For this reason, we thought it important to encourage children to find problems for themselves that would keep them practicing number facts and mathematical reasoning. Second, *everyday* means nonformal, situated in the activities of everyday life. It is important that children come to view mathematics as something that can be found everywhere, not just in school, not just in formal notations, not just in problems posed by a teacher. We wanted to get children in the habit of noticing quantitative and other pattern relationships wherever they are and of posing questions for themselves about those relationships. Two aspects of the program represent efforts to instantiate this principle. First, the problems posed in class are drawn from things children know about and are actually involved in. Second, homework projects are designed so that they use the events and objects of children's home lives: for example, finding as many sets of four things as possible in the home; counting fingers and toes of family members; recording numbers and types of things removed from a grocery bag after a shopping trip. From child and parent reports, there is good, although informal, evidence that this strategy works. Children in the program are noticing numbers and relationships and setting problems for themselves in the course of their everyday activities. Figure 6 shows part of a letter from a parent to the teacher, sharing a story of a child's everyday math engagement.

***6. Talk about mathematics, don't just do arithmetic.*** Discussion and argument are essential to creating a culture of critical thought. To encourage this talk, our program uses a combination of whole-class, teacher-led discussion and structured small-group activity by the children. In a typical daily lesson, a single relatively complex problem is presented on the chalkboard. The first phase is a class discussion of what the problem means—what kind of information is given, what is to be discovered, what possible methods of solution there are, and the like. In the second phase, teams of children work together on solving the problem, using drawings, manipulatives, and role playing to support their discussions and solutions. The teams are responsible not only for developing a solution to the problem, but also for being able to explain why their solution is mathematically and practically appropriate.

Dear Mrs Bill,

As a parent, I must share this with you

On Saturday, Raymond, Jamella and I were having lunch at Monroeville Mall. Raymond and Jamella love to spend their own money, so I told both of them that I would pay for half of their lunch. I asked Ray if he remembered what he had paid for his hot dog, fries and drink. He said "Yes, $4.33." My reply was, "O.K., tell me what I owe you. What is half of $4.33?"

He leaned back in his chair and you could just look into his eyes and see him concentrating. On the other hand, Jamella was doing the calculations on a make-believe piece of paper on the table. (This is what I usually do). Not Raymond, Mrs. Bill, he was feeling and thinking the math problem. His answer was, "$2.15, Mommy." Well, I almost fell over into my Chinese food. When I told him that he was correct, he just beamed.

**FIGURE 6 EXCERPT OF A LETTER FROM A PARENT**

The following transcript of a four-minute segment of a third-grade team's conversation as they work independently on a problem, shows how linguistic interpretation and development of manipulative displays interact in the children's work.

Mick, Joe, Anna, and Ms. B. were working on the following story problem:

> Mr. Bill bought 3 boxes of Ninja Turtle cookies for $3.79. One box costs $1.50 at other stores. Which is the better buy?
>
> How much are the $3.79 Ninja Turtles per box?

Ms. B.: I want to discuss it with your groups. I want you to show how you figured it out. And when you have it, raise your hand. I'll let you put it on. If you need manipulatives, you may just get them.

Ms. B. circulates around the room while children work at solving the problem in their respective discussion teams.

Joe: Four dollars and that's automatically over.

Anna: So here's the three boxes. [Anna puts three pieces of colored paper on the desk]

Joe: Now it's time to . . . now it's time to . . . Wait, wait a minute.

Mick: What . . .

Anna: What kind of problem could we do?

Mick: We could say, we could say three dollars and seventy-nine cents. Okay, three dollars and seventy-nine cents divided by the three boxes, because we're taking the three seventy-nine and trying to see how much each box would cost if it wasn't in a bulk. [Ms. B. appears at group table carrying the three-box unit of Ninja Turtle cookies]

Joe: All right.

Anna: I agree, I agree because we have three seventy-nine in three boxes . . . . Ms. B. brought it for second grade. Third grade will divided it up . . . in into and divided it up for second grade and third grade class.

Joe: All right, now.

Anna: So I agree.

Joe: All right, now. [inaudible] What's over three dollars [writing in notebook]

Mick: I agree.

Anna: I agree.

Joe: I agree with myself. [all three students writing in notebooks] We have to show [three dollars divided by three]. We have to put the date.

Anna: I agree. I agree . . . three dollars divided by three.

Joe: We have to show this [Joe stands and reaches into the manipulatives bin which contains bundles of 10 and 100 popsicle sticks, as well as single popsicle sticks]

Anna: How can we show this, Joe?

Mick: You could say . . . .

Joe: Three dollars. These are our three dollars. [puts down three bundles of hundred and writes something in his notebook]

Mick: So what is this, Anna, three dollars or three pennies?

Anna: Three pennies.

Mick: Okay, so three, so what do we do with this three dollars?

Anna: We divide it three hundred. [Anna picks up a bundle of one-hundred and begins to take off the rubber band]

Mick: Wait a minute . . .

Joe: We have the other two hundred.

Mick: Yeah, so . . . but are we taking off the rubber band? [addressing Anna]

Anna: Yeah, we have to.

Joe: No, we don't. Here are two more. One, two, three. [picks up and puts down the three bundles]

Anna: One goes here, one goes here, and one goes there. [puts bundles of one hundred, one at a time, on top of the pieces of colored paper]

In the third phase of the lesson, teams successively present their solutions and justifications to the whole class, and the teacher records these on the chalkboard. The teacher presses for explanations and challenges those that are incomplete or incorrect; other children join in the challenges or attempt to help by expanding the presented argument. By the end of the class period, multiple solutions to the problem, along with their justifications (as in Figures 2 and 3), have been considered, and there is frequently discussion of why several different solutions could all work, or why certain ones are better than others. In all these discussions, children are permitted to express themselves in ordinary language. Mathematical language and precision are deliberately not demanded in the oral discussion. However, the equation representations that the teacher and children write to summarize oral arguments provide a mathematically precise public record, thus linking everyday language to mathematical language (as in Figure 4).

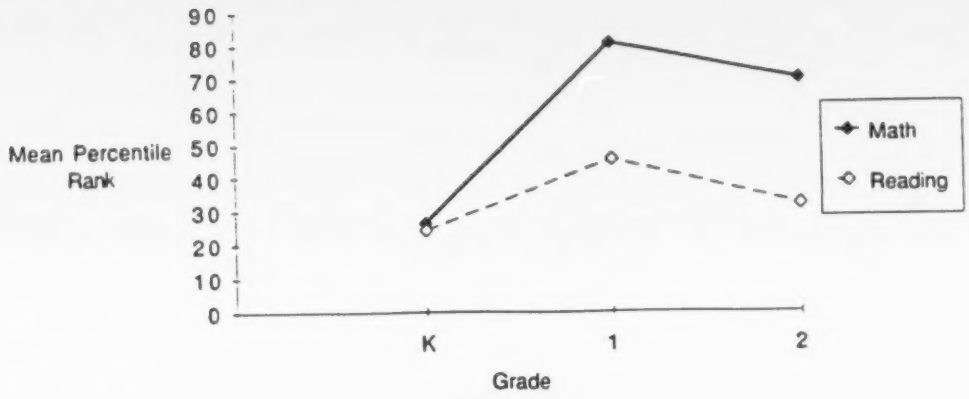
## Results of the Program

We are describing here a program that has been under development for a little over two years. The project began not as a research project but as an effort to help an ambitious teacher apply research findings to improve her teaching. During the developmental period, we did not want to impose testing programs beyond those that the school regularly administered. We are thus limited, in this period of the project's life, to data from the school's standardized testing program and from clinical interviews that we conducted with some of the children, along with some impressionistic reports of child and parent reactions to the overall program.

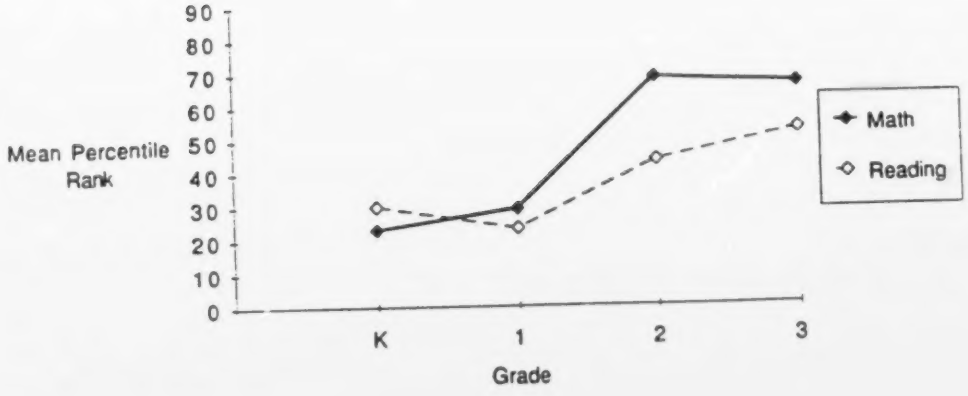
Formal evaluation data consist of scores from the California Achievement Test (CAT), which is administered in the school each September. First-graders were tested at the beginning of second grade, second-graders at the beginning of third grade. Scores on the Metropolitan Reading Readiness Test, administered by the school in March of the kindergarten year, provide data on children's general academic level before entering first grade. We have data on two cohorts of children who participated in the program, one beginning in first grade, one beginning in second grade. Figure 7a shows three years of reading and math data for Cohort A, who began the program in first grade. The children were low performers (about the 25th percentile) in both math and reading in kindergarten and remained quite low in reading in grades 1 and 2. However, their math scores rose dramatically, to a mean of the 80th percentile and stayed high (mean of 70th percentile) during the second year of the program. Figure 7b shows four years of data for Cohort B, who began the program during second grade. Like Cohort A, they were low scorers before the program. When the program was introduced in second grade, their math scores jumped to nearly the 70th percentile on average and stayed in that range through third grade. For this cohort, reading scores also rose somewhat. Reading was taught by a different teacher in the school. We are now investigating what might have been responsible for this gain. For comparison, Figure 7c shows three years of data for a cohort of children taught by the intervention program teacher before she adopted the new program. Throughout the period, mean scores remained at a low 40th to 45th percentile. An important point, one that cannot be seen in the means of the graphs, is that the math gains were not limited to only a few of the children. In Cohort A, for example, the *lowest*-scoring child at the end of the first grade was at the 66th percentile. Thus, the program appeared effective for children of all ability levels.

These global data tell only part of the story, of course. We would like to know much more for which systematic data are not yet available. Nevertheless, we can point to some indicators based on our interviews, class observations, and reports from the school. We interviewed all first-graders three times during the year, focusing on their knowledge of counting and addition and subtraction facts, along with their methods for calculating and their understanding of the principles of commutativity, conservation, and the complementarity of addition and subtraction. At the outset, these children, as

7a. Intervention Cohort A




7b. Intervention Cohort B




7c. Control Cohort

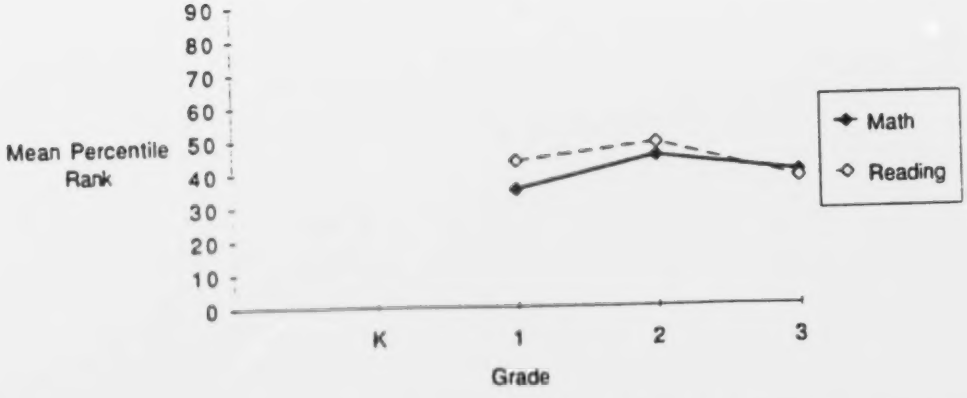



**FIGURE 7 CALIFORNIA ACHIEVEMENT TEST SCORES**

might be expected given their socioeconomic status and their parents' generally low educational background, were not highly proficient. Only one-third of them could count orally to 100 or beyond, and most were unable to count reliably across decade boundaries (e.g., 29-30, 59-60). The size of the sets that they could quantify by counting ranged from 6 to 20. About one-third of these children could not solve small-number addition problems, even with manipulatives or finger counting and plenty of encouraging support from the interviewer. Only about six appeared able to perform simple subtractions using counting procedures. Thus, these children seemed very weak in entering arithmetic knowledge, especially compared with data presented by a number of investigators for middle-class and educationally favored populations. By December the picture was sharply different. All but a handful of children were performing both addition and subtraction problems successfully, and all of these demonstrated knowledge of the commutativity of addition. At least half also were using invented procedures, such as counting on from the larger of two addends, or using procedures that showed that they understood principles of complementarity of addition and subtraction. By the end of the school year, all children were performing in this way, and many were successfully solving and explaining multidigit problems.

The following additional evidence indicates that the program was having many of the desired effects. The children displayed various examples of confidence in doing mathematical work. Many sang to themselves as they took the standardized test. When visitors came to the classroom, they would offer to show off by solving math problems. They frequently asked for harder problems. These displays came from children of almost all ability levels. They had not been typical of any except the most able children the preceding year. Homework was more regularly turned in than in preceding years, without nagging or pressure from the teacher. Children often asked for extra math periods. Many parents reported that their children loved math and wanted to do math all the time. Parents also sent to school examples of problems that children had solved on their own in some everyday family situation. Knowing that the teacher frequently used such problems in class, parents asked that their child's problems be used. It is notable that this kind of parent engagement occurred in a population of parents that is traditionally alienated from the school and tends not to interact with teachers or school officials.

## Conclusion

We believe we have made a promising start at reaching our goals. We have shown that an interpretation- and discussion-oriented mathematics program can begin at the outset of school by building on the intuitive mathematical knowledge that children have as they enter school. Our standardized test score data show that this kind of thinking-based program also succeeds in teaching the basic number facts and arithmetic procedures that are the core of the traditional primary mathematics program. It is not

necessary to teach facts and skills first and only then go on to thinking and reasoning. The two can be developed simultaneously. Assuming that we can maintain and replicate our results, this means that an interpretation- and discussion-oriented program can serve as the basic program in arithmetic, not just as an adjunct to a more traditional knowledge and skills curriculum.

Moreover, our results show that an interpretation-oriented mathematics program can be suitable even for children who are not socially favored or, initially, educationally able. The children with whom we have worked come disproportionately from among the least favored of American families. Many are considered to be educationally at risk; their educational prognosis, without special interventions or changed educational programs, is poor. Yet these children learned effectively in a type of program that, if present in schools at all, has been reserved for children judged able and talented—most often those from favored social groups.

What is at issue here, as we suggested at the outset, is not only an apparently successful program but also some fundamental challenges to dominant assumptions about learning and schooling. As we worked to develop this program, we realized that a new theoretical direction was increasingly dominating our thinking about the nature of development, learning, and schooling. This is the view, shared by a growing minority of thinkers in the various disciplines that comprise cognitive science, that human mental functioning must be understood as fundamentally situation-specific and context-dependent, rather than as a collection of context-free abilities and knowledge. This apparently simple shift in perspective in fact entails reconsideration of a number of long-held assumptions in both psychology and education.

Until recently, educators and scholars have defined the educational task as one of teaching specific knowledge and skills. As concern has shifted from routine to higher-order or thinking abilities, we have developed more complex definitions of the skills to be acquired and even introduced various concepts of *meta* skill in the search for teachable general abilities. But we have continued to think of our major concern as one of identifying and analyzing particular skills of reasoning and thinking and then finding ways to teach them, on the assumption that successful students then will be able to apply these skills in a wide range of situations.

As we developed our program, we found ourselves less and less asking what constitutes mathematics competence or ability for young schoolchildren, and more and more analyzing the features of the mathematics classroom that provide activities that exercise reasoning skills. This meant choosing story problems on the basis of the mathematical principles they might illustrate and developing forms of classroom conversation designed to evoke public reasoning about these principles. Our focus on mathematics as a form of cultural practice did not deny that children engaging in mathematical activity must be knowledgeable and skillful in many ways. However, our emerging perspective led us to focus far less on the design of "lessons" than on the

development of a sequence of problem-solving situations in which children could successfully participate. Another way of saying this is that we were trying to create an *apprenticeship* environment for mathematical thinking in which children could participate daily. We expected them to acquire thereby not only the skills and knowledge that expert mathematical reasoners possess, but also a social identity as a person who is able to and expected to engage in such reasoning (see Lave, in press).

Our program constitutes a version of the *cognitive apprenticeship* called for by Collins, Brown, and Newman (1989) in a recent influential paper. Its very success, however, calls into question some aspects of the apprenticeship metaphor as applied to early learning in a school environment. Among these is the nature of the master–apprentice relationship. In traditional apprenticeship, apprentices seek to become like their masters, and masters continually display all elements of skilled productive activity in their field of expertise. Teaching is only a secondary function of the traditional master. This simple—indeed, perhaps oversimplified—relationship does not seem applicable to the school setting, where the teacher's predominant function is not to *do* mathematics but to *teach* it. We will need to work out the particular role of the teacher in designing an environment *specifically for learning purposes.* A second issue surrounding cognitive apprenticeship in school is how to ensure that necessary particular skills will be acquired, even though the daily focus of activity is on problem solving and reasoning. Our first-year standardized test results suggest that we have not done badly on this criterion, but we need to understand better than we do now just *what* it is in our program that has succeeded and what the limits of our methods might be. In short, we offer this paper as only a very preliminary report on what we expect to be a long-term effort to revise instructional practice in ways that will bring educators closer to being able to meet the goal of shaping dispositions and skills for thinking through a form of socialization into cultural environments that value and practice thinking.

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# DISCUSSION: EXPLORING A REASONING-BASED PROGRAM FOR TEACHING ARITHMETIC[1]

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In their paper for this volume, Resnick, Bill, Lesgold, and Leer describe a mathematics program for young children that they believe is especially good for those who are at risk of school failure. We have been asked to provide comments on their paper. As coauthors, we bring to bear on the task several different perspectives and types of expertise. We bring the expertise of a classroom teacher with many years of experience teaching mathematics to young disadvantaged children. We also bring the expertise of several cognitive psychologists specializing in ways to enhance learning and instruction, especially through the use of technology.

We want to begin by briefly underscoring two basic features of Resnick and colleagues' program that make it especially timely. First, the program was developed and tested in classrooms made up of children from disadvantaged backgrounds with records of poor performance in traditional academic settings. There is a current, critical need for instruction that will be effective for children such as these who are at risk of school failure. Recent estimates warn that at least 30% of elementary and secondary students in the United States are educationally at risk, and that this proportion may rise rapidly in the future.

Second, the program represents a shift away from the format of traditional remedial programs. Resnick and colleagues do not emphasize drill and practice in "the basics," nor do they attempt to simplify the content that is taught. Instead, their program is designed to help children recognize the power of their own mathematical intuitions and to introduce them to sophisticated ideas about mathematical concepts and procedures. The value of teaching skills and knowledge in the context of meaningful, practical activities, rather than as decontextualized exercises, has only recently received much attention by educational researchers (e.g., Brown, Collins & Duguid, 1989; Cognition and Technology Group at Vanderbilt, 1990; Resnick & Klopfer, 1989). To our knowledge, this is one of the first systematic efforts to teach mathematics to children of this age and skill level using a reasoning-based approach, and we applaud the efforts of Resnick and colleagues in their endeavor.

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the authors' program attempts to draw on children's informal mathematical knowledge is by encouraging them to use their informal mathematical knowledge as a basis for solving problems in mathematics class. Many times the message that we give children, unintentionally or not, is that mathematics consists only of the formal concepts and algorithms explicitly taught in mathematics class. We unintentionally teach children to devalue their intuitive knowledge, or, worse yet, to believe that formal and informal mathematical knowledge bear no relation to one another. Resnick et al.'s program encourages children to trust their mathematical intuitions by encouraging them to solve problems in any way they are able, and not merely by means of standard algorithms.

Another major instructional principle is to introduce children to formal mathematical content as early as possible. Guided by this principle, Resnick et al. introduce some content much earlier and in a different sequence than is done traditionally (e.g., addition and subtraction with regrouping in first grade). Their goal is to equip students with more powerful ways to think mathematically, that is, to quantify information and link it with formal expressions and operations. The assumption is that children are ready for this content if they are helped to map it onto their protoquantitative schemas, and if they are provided with appropriate manipulatives and with opportunities for extended practice. As Resnick et al. note, their nontraditional approach discourages the teaching of isolated skills and enables children to draw more fully on their existing knowledge.

From our perspective, Resnick et al.'s strategy of selecting curricular content based on what is known about children's knowledge at different ages is extremely valuable. There is ample evidence to suggest that tailoring instruction to students' current knowledge, formal or informal, is a critical factor in determining whether and how much students learn (Anderson & Smith, 1984; Bransford & Vye, 1989; Carey 1986; Feltovich, Spiro, & Coulson, 1988).

The authors' commitment to creating a curriculum that is sensitive to children's mathematical development leads us to question whether similar concern is given to other areas of children's cognitive and social/emotional development. For example, to what extent does the curriculum presuppose a particular level of language and social development? We raise the issue of children's language skills in part because an important part of the curriculum involves having children engage in dialogue related to mathematics concepts and problems, and in part because at-risk students are very often language-delayed. We have a similar question regarding the level of social development presupposed by the program. Once again, we raise the issue because the program incorporates group discussion where students need to be able to reflect on and respond to other students' comments in ways that promote learning.

### Cognitive Apprenticeships in Mathematics

Another theoretical influence that can be discerned in Resnick et al.'s program is work on "situated learning" (Brown, Collins, & Duguid, 1989; Cognition and Technology Group at Vanderbilt, 1990; Rogoff & Lave, 1984). According to this approach, one of the major "failings" of schools is that the kinds of activities and tasks comprising various school subjects (i.e., history, mathematics, and so forth) tend to be arbitrary and unrelated to the activities engaged in by experts and professionals who work in these domains. For example, students of mathematics are likely to spend their time computing answers to already formulated problems, whereas the kind of mathematical thinking that an expert mathematician might engage in would more likely involve inventing mathematical procedures, exploring mathematical relations, and so forth (see Schoenfeld, 1989, for more detailed illustrations of authentic mathematics tasks). Proponents of this approach also argue that schools tend to emphasize abstract concepts and knowledge and often do not help students understand how and when to apply this knowledge. As a result, this knowledge is not used to help solve problems in contexts other than the particular school context in which it was originally learned.

Resnick et al.'s curriculum is consistent with a cognitive apprenticeship approach, and aspects of the curriculum are reminiscent of other curricula for older children based on the same approach (e.g., Lampert, 1986; Schoenfeld, 1989). The curricular activities described in the paper are very different from the activities of traditional mathematics classes and are very like the kinds of cognitive activities in which expert mathematicians take part. There is an emphasis on problem solving and on inventing mathematical procedures. Students are also encouraged to think about mathematics as something that is found outside of school; they are encouraged to notice quantitative and other pattern relationships in everyday situations and to notice everyday situations where mathematics is used to solve problems.

We believe that by helping students construct the meanings of mathematics from their own perspectives, the curricula will be more effective than traditional curricula. Indeed, Resnick et al.'s initial evaluation data suggest that students better understand basic mathematical concepts and procedures. We also suspect that students will be better able to apply what they have learned to other situations, both in and out of the classroom, partly because students are given ample opportunity to reflect on quantitative concepts and procedures and partly because they practice this knowledge in authentic problem situations.

### The Curriculum from the Perspective of Classroom Teachers

From the perspective of the classroom teacher, Resnick et al.'s program presents some exciting possibilities. The program illustrates ways in which teachers can help

disadvantaged children learn to reason better and at the same time acquire a deep understanding of quantitative concepts and operations. A key to the program's success is its emphasis on mapping new knowledge onto the skills and experiences of children. A major difficulty that teachers of disadvantaged children face is that their children's range of experience and their familiarity with school-like tasks and skills are often limited. The program provides some excellent examples of how to create meaningful problem solving by carefully selecting familiar problem situations, by helping children notice mathematical situations in their everyday lives, and by motivating parents to teach their children. Teachers will be especially interested to learn more about how the program stimulated parental involvement, since many disadvantaged children come from dysfunctional families or from families where the parents have little education, and where there is often limited contact with school personnel.

Although the program presents exciting possibilities, it is likely to present some additional challenges as well. The teacher's role in Resnick et al.'s program differs in major ways from the teacher's role in the typical arithmetic classroom; as a result, teachers will have questions—and in some cases reservations—about the program. For instance, the program involves extensive use of small-group and teacher-led discussion. Many teachers will have questions about whether the program will work in their classrooms because of their children's limited language skills. In thinking about disseminating their program, it will be important for Resnick and colleagues to address the question of whether some minimum language competency is required, and what strategies can help teachers circumvent problems that may arise in class because of language deficiencies. Another challenge of the program is that many classroom teachers rarely if ever use small-group activities with young disadvantaged children. Because of the belief that disadvantaged children tend to exhibit behavior problems, some teachers may be afraid to try small groups for fear of losing control of the class. These teachers will need special reassurances. In general, teachers will be interested in learning how to facilitate social interactions when children work in groups.

In addition to concerns about whether their children possess the requisite competencies and about the teaching strategies to use to obviate problems that these deficiencies create, we suspect that many teachers will feel that they need specialized skills training before they would be able to implement the program in their classrooms. First, we think that teachers will want to learn techniques for conducting small groups. We assume that when children first work in small groups, they will need a great deal of support from teachers, and that teachers will have to model the kinds of interactions that they want children to engage in. Second, teachers will also need training in techniques that help children create procedures for solving problems. Teachers are more likely to be used to teaching one "right" procedure for solving a class of problems; therefore, they may have difficulty knowing how to help children invent their own procedures for solving problems. Finally, as surprising as it may seem, many teachers may also need special training in how to teach mathematics in a way that helps children understand the number

system. Sadly, mathematics instruction frequently emphasizes drill and practice on procedures, with little attention to comprehending the underlying basis for these procedures, and many teachers do not use manipulatives as a comprehension tool. These teachers will need special guidance in transforming their instruction, particularly as it relates to helping children understand how mathematical operations such as addition, subtraction, multiplication, and division relate to the base 10 system. We think that teaching teachers how to use a comprehension-based approach to instruction, particularly as this relates to the use of manipulatives, is especially important for reasons that we describe in greater detail below.

## A Curriculum for Building Mental Models

### What Is a Mental Model and Why Is It Useful?

We now turn our attention to discussions of particular, cognitive aspects of problem solving, and how the program that Resnick and colleagues describe may be especially successful in enabling children to gain a deeper understanding of mathematics. We think that two of the more effective components of Resnick et al.'s program, especially for children who are at risk for school failure, are their extensive use of manipulatives and their use of familiar contexts for problem solving. Our view arises from recent theories that conceive of comprehension and problem-solving as tasks that require students to construct "mental models." A mental model is similar to physical models or pictures of specific situations. In fact, for the purposes of this discussion, mental models can be thought of as mental pictures.

Why is it important that a child be able to construct a mental model or a mental picture of a problem? Because to construct a mental model, a child has to be clear on what the objects in the situation are like, how they are arranged, and what relationships they have to each other. Unless the child has a mental model of what the situation is, he or she will not know how to manipulate the elements of the situation in order to come up with a solution. For example, in the "barrette" problem that Resnick et al. describe, students need to form a mental model that specifies that (a) there are four main elements (i.e., the bags); (b) inside each of those four bags are nine items (i.e., the barrettes); and (c) the total number of all the barrettes in all the bags need to be combined and compared with the number *95*. By having a mental model of the situation, the child can understand why it makes more sense to combine four groups of *9* together than to add the numbers *4* and *9*. The correct operation is the one that makes sense in that situation. In short, the notion of using a mental model to understand and solve a problem is very different from the notion of using a set of "rules," such as "*and* means to add," and "*more than* means to subtract."

### How the Program Supports Mental Model Construction

Just as the creation of a picture or a physical model takes physical effort, so the creation of a mental model or image takes mental effort. For example, an image of four bags of barrettes is very different from how the words of the story problem sound or look on a page, so students need to mentally translate the words in the problem into elements in a mental model that are correctly arranged. This translation process may require considerable mental effort for the young child, and the amount of mental resources required may exceed the limits of the child's pool of memory resources.

This analysis suggests how physical aids like manipulatives, pictures, and "acting out situations" can help problem solving. Instead of having to perform all the translations from words to mental models "in the head," the child can use these physical aids to help keep track of what the elements in the problem are and how they are arranged. Moreover, the use of familiar objects and contexts is helpful because the child can more easily image familiar objects and can draw on memories of past familiar scenes for information on how the mental model should look.

As children become practiced in translating word problems to mental models, we would hope that the translation process would require less and less effort, so that eventually children are able to construct the mental model from the text and numbers in the problem alone. Whether a reliance on physical aids is something that will automatically diminish, or whether children need to be systematically taught to construct mental models under less supportive conditions in which physical aids are not available, is a question that is likely to be answered in further research using Resnick et al.'s approach.

### The Generality and Transfer of Mental Model Skills

The construction of mental models from text is something that is necessary not just for understanding story problems; recent research suggests that good readers construct mental models for all types of text, and it is the construction of mental models that leads to improved comprehension and memory of text information (Glenberg, Meyer, & Lindem, 1987; Johnson-Laird, 1983; Sharp, 1991; Sharp & McNamara, 1991). Therefore, if children learn in a math program how to translate text information into a mental model, then we would hope that they would use this skill for understanding other kinds of reading material. We note that the reading scores of one of the groups in Resnick et al.'s program did appear to improve, whereas the reading scores of the other group did not. The authors mention that they are investigating why the reading scores improved for one of their groups, but we would suggest that they might turn the question around and investigate why the reading scores of the other group did *not* improve. Research has shown that students often fail to transfer skills that they learn in one content domain to other content domains, so if the skills that students learn in the math

program are in fact applicable to reading, then students may need to be explicitly taught to transfer these skills from the math class to the reading class.

### Videodisc Environments for Building Mental Models

In this section, we describe how videodisc environments may be used to achieve some of the same instructional goals that Resnick and colleagues describe, and also why we think that videodiscs may provide a means for expanding these goals. Like Resnick and her colleagues, we think that children should learn math in the context of real-world problems, and that they should learn how to reason about and discuss mathematical solutions. In the previous sections, we described our belief that the key to enabling children to reason about math is to provide them with the tools for building rich mental models of problem situations. We now will discuss why we think that videodiscs can be extremely effective in providing those tools.

For example, one of our projects at the Cognition and Technology Group is a videodisc mathematics problem-solving series called *The Adventures of Jasper Woodbury* (Cognition and Technology Group at Vanderbilt, 1990). Although the series is designed for students who are several years older than Resnick et al.'s—the series is for fifth- and sixth-graders—we are developing materials for younger students. Each disc in the series is an adventure story. At the end of the story, the main character is faced with a complex problem whose solution involves mathematics. For example, in one video the main character, Emily, is faced with the problem of rescuing a wounded eagle. The eagle is stranded in a remote area where the nearest road is several miles away. To rescue the eagle, Emily must decide which of several different routes to take and what means of transportation to use (i.e., ultralight, foot, truck, or some combination of means). Each route must be evaluated in turn with respect to the distance to be traveled, the speed and range of the selected vehicle, and, of course, estimated trip time. All the mathematical information needed to solve Emily's problem is in the story. To reach a solution, students must formulate various plans for rescuing the eagle, decide how to evaluate each plan mathematically, search the video for the relevant mathematical information, and perform computations using this information.

Our research using the *Jasper* series, as well as other videodisc-based materials, has shown that students learn more from the video instruction than from traditional instruction on the same content. For example, *Jasper*-taught students are more adept at the higher-order reasoning skills involved in complex mathematical problem solving than students who receive traditional instruction and practice on story problems. They are better able to generate and logically evaluate plans for solving complex problems, and they are also more successful at solving them.

We believe that the videos directly provide students with rich mental models of problem situations, and in this way help them bypass problems that they may experience

in constructing mental models based on verbal or written scenarios of problems (e.g., a written story problem). The videos also provide extensive background knowledge, without which many students, particularly at-risk students, would not be able to form mental representations of problems they are given. Mental models created from video thus provide the scaffold for lower-level skills that enables children to practice higher-level problem-solving skills.

Video problem solving seems to have important effects on motivation as well. Teachers consistently observe a high degree of enthusiasm and on-task behavior among students when using *Jasper.* Furthermore, they see this in Chapter 1 and high-achieving classes alike. Chapter 1 students are positive about this program despite the fact that in general they tend to have relatively negative attitudes toward mathematics. At this point we can only speculate on what accounts for the increased motivation that we have observed. Novelty may play a role, and the use of video—a very comfortable medium for children—may also be a factor. But we also think that a very important factor is that students come to understand that mathematics has a function outside the classroom in helping solve problems from everyday life. In this way, the videos help forge what Resnick et al. suggest is an important link between everyday experience and formal mathematics.

We think that children in Resnick et al.'s program might benefit from problem-solving instruction where some of the problems are depicted in video format. Of course, one could argue that the best way to practice problem solving is in the context of everyday problems that students themselves identify. And in fact, one principle of Resnick et al.'s program is to encourage finding everyday problems. Although we would agree that everyday problems are the ultimate "authentic" context and that it is important to encourage children in this way, we would argue that they have instructional limitations. For example, problems that are part of one child's experience are not necessarily part of other children's experiences, making them less effective for instruction. This is where video has an important advantage. Because video can be taken into the classroom, all children can share the experience. We would also argue that everyday problems that children identify are likely to be less useful than video because they are apt to be problems that are already within children's competence levels. In this way, finding everyday problems may be most effective as a vehicle for having children *practice* mathematical skills. Video problem-solving environments, on the other hand, can also be used to extend children's mathematical competence into new skill areas.

Another way in which Resnick et al.'s program might benefit from the introduction of videodisc problem solving is in the quality of the classroom discussion among students and between teacher and students. We have found that students initiate more questions and that there is more sustained discussion among students when they are working on video-based tasks. Note that this is very different from typical classroom talk, which consists of a repetitive cycle where the teacher asks a question and a student responds.

In part we think this higher-quality discussion occurs because all students share a rich understanding, or what we have been calling a mental model, of the problem situation, which enables more students to take an active part in classroom activities. Video problem-solving contexts may prove especially effective in reasoning-based programs such as Resnick et al.'s, where dialogue among students is an essential vehicle for learning.

## Summary of Comments

The curriculum described by Resnick, Bill , Lesgold, and Leer meets two important educational needs: the need for more early elementary programs that help disadvantaged children and the need for more programs that teach thinking skills to these children. We think the authors have developed a program that will have an important impact on young children's mathematical thinking. Our goal has been to acknowledge the curriculum's strong theory base and to discuss specific cognitive mechanisms that may mediate students' learning. At the same time, we raised questions concerning the language and social skills presupposed by the program, and the new roles required of teachers. We also have suggested ways to enhance instruction in everyday problem solving by incorporating videodisc problem-solving contexts. We believe that video provides the scaffold for lower-level skills that enables children to practice higher-level problem solving in authentic contexts.

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